

THE WARING WIDTHS OF POLYNOMIALS ON MATRIX ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT

We study Waring-type problems for polynomial maps on matrix algebras over algebraically closed fields. For a nonconstant polynomial, we investigate additive and multiplicative representations of matrices as sums and products of polynomial values. Consequently, the corresponding additive and multiplicative Waring widths are at most two.

Keywords: Matrix algebra; polynomial; Waring width

1. Introduction and literature review

The classical Waring problem, formulated by Edward Waring in 1770, concerns the existence of a positive integer N such that every positive integer b can be represented as a sum of N n th powers of nonnegative integers. We recall the problem in the following form.

Problem 1. For each integer $d \geq 2$, determine the least positive integer $N(d)$ such that every positive integer n can be represented in the form

$$n = a_1^d + a_2^d + \cdots + a_{N(d)}^d$$

where a_i 's are positive integers. The integer $N(d)$ is called the Waring width of d .

In 1909, David Hilbert established that such an N indeed exists for every positive integer n . The least possible value of N is traditionally denoted by $g(n)$. For instance, Lagrange's four-square theorem gives $g(2) = 4$, while several other values have also been determined, including $g(3) = 9$, $g(4) = 19$, and $g(5) = 37$ [1].

Despite these known cases, the

function $g(n)$ is still not completely understood. Moreover, the Waring problem has generated a broad range of related questions that remain the subject of ongoing research in Number Theory. These developments have, in turn, motivated the study of Waring-type questions in more general algebraic settings. One natural way to obtain such extensions is to replace the n th powers appearing in the classical problem with more general algebraic expressions, leading to a variety of analogues across different algebraic structures.

For a comprehensive account of Waring-type questions in groups, Lie algebras, and associative algebras, we refer the reader to the survey of Brešar and Martinez [2]. The present work is concerned with analogous questions for matrix algebras.

Let F be a field, let $m, n \geq 1$, and let $f \in F[x_1, \dots, x_m]$. We consider two natural matrix-algebraic analogues of the Waring problem. In the additive setting, we seek the smallest positive

integer r for which every matrix in $M_n(F)$ can be expressed as a sum of r values of f . More precisely, we say that f has additive Waring width r on $M_n(F)$ if r is the least positive integer such that, for every $A \in M_n(F)$, there exist $B_{ij} \in M_n(F)$ satisfying

$$A = f(B_{11}, \dots, B_{1m}) + \dots + f(B_{r1}, \dots, B_{rm}).$$

The multiplicative counterpart is defined in a similar manner. Namely, f is said to have multiplicative Waring width r on $M_n(F)$ if r is the least positive integer such that every $A \in M_n(F)$ admits a factorization

$$A = f(B_{11}, \dots, B_{1m}) \cdots f(B_{r1}, \dots, B_{rm}),$$

where all B_{ij} belong to $M_n(F)$.

The main result of this paper establishes that, over an algebraically closed field, both the additive and multiplicative representations can be obtained with just two polynomial values. More precisely, we prove the following theorem.

Theorem 2. Let F be an algebraically closed field, let $n, m \geq 1$ be positive integers, and let $f_1, f_2, f_3, f_4 \in F[x_1, \dots, x_m]$. Then, for every $A \in M_n(F)$, there exist m -tuples of matrices $B, C, D, E \in M_n(F)^m$ such that $A = f_1(B) + f_2(C) = f_3(D)f_4(E)$. In particular, both additive and multiplicative Waring width of any polynomial $f \in F[x_1, \dots, x_m]$ on $M_n(F)$ are at most 2.

2. The proof of Theorem 2

The following statement can be viewed as a special case of results

established by Botha; see Theorem 2.6 in [3], Theorem 2.1 in [4], Theorem 2.2 in [5].

Theorem 3. Let F be an algebraically closed field and let $n \geq 1$. Then every matrix in $M_n(F)$ admits both an additive and a multiplicative decomposition into two diagonalizable matrices in $M_n(F)$.

Theorem 3 naturally leads to two basic questions concerning polynomial maps on algebras. First, how does the value of a polynomial behave under conjugation? Second, what happens when the variables are evaluated at diagonal matrices? We address these two questions separately.

We begin with invariance under conjugation. Let R be a commutative unital ring and let A be a unital R -algebra. Consider the polynomial

$f = x_1x_2 + x_2x_3 + x_1^2 \in R[x_1, x_2, x_3]$. For arbitrary $a_1, a_2, a_3 \in A$, put $u = f(a_1, a_2, a_3)$. If $s \in A$ is invertible and we conjugate each of the variables by s , setting $b_i = s^{-1}a_is$ where $i = 1, 2, 3$, then $b_1b_2 = s^{-1}a_1a_2s$, and, similarly, $b_2b_3 = s^{-1}a_2a_3s$, and $b_1^2 = s^{-1}a_1^2s$. Consequently, $f(b_1, b_2, b_3) = s^{-1}us$.

Thus, once an element occurs as a value of f , every element conjugate to it occurs as a value of the same polynomial. This observation is not specific to the polynomial considered above; it holds for arbitrary polynomials over a commutative ring.

Theorem 4. Let R be a commutative unital ring, let A be a unital

R -algebra, and let $f \in R[x_1, \dots, x_m]$ be a polynomial. Let $f': A^m \rightarrow A$ denote the polynomial map induced by f . Then $Im(f')$ is invariant under conjugation.

Here the polynomial map induced by f is the map from A to A that sends each element a to the element $f(a)$, obtained by evaluating f at a .

The proof of Theorem 4 follows the same argument as in the example above and is therefore omitted for brevity.

Now, we continue with evaluation on diagonal matrices. It is not difficult to verify the following theorem.

Theorem 5. Let R be a commutative unital ring and let $f \in R[x_1, \dots, x_m]$. For each $i = 1, \dots, m$, let $D_i = diag(d_{i1}, d_{i2}, \dots, d_{in}) \in M_n(R)$.

Then,

$$f(D_1, \dots, D_m) = (f(d_{11}, \dots, d_{m1}), \dots, f(d_{1n}, \dots, d_{mn})).$$

In other words, evaluating a polynomial at diagonal matrices preserves diagonality. Moreover, the entries on the main diagonal are obtained by evaluating the original polynomial coordinatewise on the corresponding diagonal entries of the matrices.

With all in hand, we are in a position to prove Theorem 2. Indeed, we first prove the additive assertion. By Theorem 3, there exist diagonalizable matrices $A_1, A_2 \in M_n(F)$ such that $A = A_1 + A_2$. Since A_1 is diagonalizable, there exist $P_1 \in GL_n(F)$ and $\lambda_1, \dots, \lambda_n \in F$ such that

$$A_1 = P_1 diag(\lambda_1, \dots, \lambda_n) P_1^{-1}.$$

Since f_1 is nonconstant and F is

algebraically closed, for each $i \in \{1, \dots, n\}$, there exists $(a_{i1}, \dots, a_{im}) \in F^m$ such that $f_1(a_{i1}, \dots, a_{im}) = \lambda_i$. By Theorem 5, there exist matrices $D_1, \dots, D_m \in M_n(F)$ such that

$$diag(\lambda_1, \dots, \lambda_n) = f(D_1, \dots, D_m).$$

Set $B_j = P_1 D_j P_1^{-1}$ where $j = 1, \dots, m$.

By Theorem 4, we then have

$$A_1 = f_1(B_1, \dots, B_m).$$

Applying the same argument to A_2 and f_2 , we obtain matrices $C_1, \dots, C_m \in M_n(F)$ such that

$$A_2 = f_2(C_1, \dots, C_m).$$

Therefore,

$$A = A_1 + A_2 = f_1(B_1, \dots, B_m) + f_2(C_1, \dots, C_m).$$

This proves the additive assertion.

For the multiplicative assertion, Theorem 3 gives diagonalizable matrices $A_1, A_2 \in M_n(F)$ such that $A = A_1 A_2$. As above, since A_1 and A_2 are diagonalizable and f_3, f_4 are nonconstant, Theorems 4 and 5 imply that there exist matrices $B_1, \dots, B_m, C_1, \dots, C_m \in M_n(F)$ such that $A_1 = f_1(B_1, \dots, B_m)$ and $A_2 = f_2(C_1, \dots, C_m)$. Consequently,

$$A = A_1 A_2 = f_1(B_1, \dots, B_m) f_2(C_1, \dots, C_m).$$

Thus, the multiplicative assertion also follows.

3. Conclusion

In this paper, we have established Waring-type results for polynomial maps on matrix algebras over algebraically closed fields. In particular, we have shown that every matrix can be expressed as the sum or the product of at

most two values of arbitrary nonconstant polynomials. Our approach is based on the decomposition of matrices into diagonalizable matrices, together with the behavior of polynomial evaluations on diagonal matrices and their invariance under conjugation.

These results naturally lead to several questions for further investigation. One particularly natural problem is to determine to what extent the algebraic closedness assumption on the underlying field can be relaxed. More precisely, one may ask:

Problem 6. Let F be a field, $n, m \geq 1$, and let $f \in F[x_1, \dots, x_m]$ be a nonconstant polynomial. Under what conditions on F, f, n , can every matrix in $M_n(F)$ be represented as a sum, or as a product, of two values of f ?

It would also be interesting to determine the corresponding Waring widths when the number of summands or factors cannot be reduced to two. These questions may provide a natural starting point for extending the present results to broader classes of fields and algebras.

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ĐỘ RỘNG WARING CỦA CÁC ĐA THỨC TRÊN ĐẠI SỐ MA TRẬN TRÊN TRƯỜNG ĐÓNG ĐẠI SỐ

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TÓM TẮT

Chúng tôi nghiên cứu các bài toán kiểu Waring đối với các ánh xạ đa thức trên các đại số ma trận trên trường đóng đại số. Với một đa thức không hằng, chúng tôi nghiên cứu các biểu diễn cộng và nhân của các ma trận dưới dạng tổng và tích của các giá trị của đa thức. Từ đó, các độ rộng Waring cộng và nhân tương ứng đều không vượt quá hai.

Từ khóa: Đại số ma trận; đa thức; độ rộng Waring